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Areas, anti-derivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts

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ABSTRACT

Research in mathematics and science education reveals a disconnect for students as they attempt to apply their mathematical knowledge to science and engineering. With this conclusion in mind, this paper investigates a particular calculus topic that is used frequently in science and engineering: the definite integral. The results of this study demonstrate that certain conceptualizations of the definite integral, including the area under a curve and the values of an anti-derivative, are limited in their ability to help students *make sense* of contextualized integrals. In contrast, the Riemann sum-based "adding up pieces" conception of the definite integral (renamed in this paper as the "multiplicatively-based summation" conception) is helpful and useful in making sense of a variety of applied integral expressions and equations. Implications for curriculum and instruction are discussed.

1. Introduction and rationale

In the last decade there has been increased attention given to researching students' understanding and use of the calculus topic of the definite integral (e.g., Bajracharya & Thompson, 2014; Jones, 2013; Kouropatov & Dreyfus, 2013; Rasslan & Tall, 2002; Sealey, 2014; Thompson & Silverman, 2008; Wemyss, Bajracharya, Thompson, & Wagner, 2011). Integration is a key topic that deserves our attention for several reasons. In a purely mathematical sense, it is a significant component in subsequent mathematics courses, including further coursework in the calculus series (Salas, Etgen, & Hille, 2006; Stewart, 2012; Thomas, Weir, & Hass, 2009) and in higher level mathematics, such as differential equations (Boyce & DiPrima, 2012) and complex analysis (Brown & Churchill, 2008). Thus, integration is an important foundational concept for a program of study in mathematics. However, the integral goes much further than this; it also serves as the basis for many real world applications in science and engineering. Physics and engineering textbooks regularly use integrals to define and compute natural phenomena like force, mass, center of mass, impulse, flux, circulation, energy, work, tension, and aspects of kinematics (Hibbeler, 2012; Pytel & Kiusalaas, 2010; Serway & Jewett, 2008; Wilson, Buffa, & Lou, 2010).

Yet this begs the question why we, as mathematics educators, should be concerned with how science disciplines, such as physics and engineering, use integration. The answer to this question is two-fold. First, as a "service course," first-year calculus at many universities is largely filled with students planning on majoring in science and engineering fields (Bressoud, Carlson, Mesa, & Rasmussen, 2013). Consequently, it may be that the students coming into our calculus classes are less motivated by pure mathematics than they are about being able to use the mathematics in their respective disciplines. Calculus instructors should be willing to address the needs of this large segment of their student population. Second, there is currently a push to

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improve science, technology, engineering, and mathematics (STEM) education as a connected whole (College Board, 2012; President's Council of Advisors on Science and Technology, 2012). This push implies that effort should be made by instructors of all STEM courses, including calculus, to assist in promoting success in these important fields of study.

Unfortunately, studies in both mathematics and science education give evidence of a serious disconnect between mathematics and the science disciplines it serves. For example, science contexts give additional layers of meaning to variables in mathematical expressions and equations, making the application of mathematics to science challenging (Dray & Manogue, 2005; Redish, 2005; Torigoe & Gladding, 2011). The way students learn mathematics in their mathematics courses does not always line up with how they need to draw on that knowledge in science and engineering (Gainsburg, 2006, 2007). Bridging that gap must involve insight from the mathematics education community; it is not a task for science education alone. We should develop mathematical understanding in our courses that supports application to other fields.

While we are beginning to learn how students create and hold knowledge about the definite integral (see Grundmeier, Hansen, & Sousa, 2006; Jones, 2013; Rasslan & Tall, 2002), there is scant evidence about how these cognitive constructs actually play out for students in making sense of the expressions and formulas in which integrals are present. Sealey and Engelke (2012) suggest that the area conception alone is not sufficient for robust understanding of integrals, and (Jones, 2013) supports this conclusion with an example of a student who struggled to interpret a physics integral through the area lens. Furthermore, Jones describes how a shift to an *adding up pieces* conception helped this student make sense of the integral he was struggling with. Neither of these studies, however, constitutes a deep analysis of how various conceptualizations of the definite integral provide students with the cognitive resources to understand integrals in mathematics and science contexts. Without such an analysis, it is difficult to determine how to provide students the opportunities to construct knowledge of the integral that will satisfactorily enable them to apply their knowledge to science and engineering.

This study attempts to provide some of this needed analysis by carefully examining how certain conceptualizations of the integral drive understanding in mathematics and science contexts. The study examines, in both of these contexts, student conceptualizations of the definite integral that are related to three common interpretations of the definite integral: (a) as the area under a curve, (b) as the values of an anti-derivative, and (c) as the limit of Riemann sums (see Salas et al., 2006; Stewart, 2012; Thomas et al., 2009). Note that even though the anti-derivative notion is often thought of as the province of *indefinite* integrals, students show a tendency to interpret *definite* integrals through the anti-derivative lens as well (Jones, 2013). Each of these three conceptualizations of the integral is evaluated for how helpful or useful it is for *making sense* of definite integral expressions and equations. Specifically, this paper attempts to shed light on the following three core questions: (1) Which of these three conceptualizations of the definite integral appear to be most useful for making sense of integrals in a pure mathematics context? (2) Which of these three conceptualizations of the definite integral appear to be most useful for making sense of integrals in an applied physics context? (3) What is the overlap or disjunction between these two contexts?

2. Background

2.1. Symbolic forms of the definite integral

This paper builds on a previous study (Jones, 2013) that details several conceptualizations of the definite integral held by calculus students. In this section, the reader is briefly acquainted with three of the conceptualizations described in that study, which deal with how students cognitively hold the familiar notions of area under a curve, anti-derivatives, and Riemann sums in connection with integrals. While there are certainly meanings beyond these three conceptions (e.g., Hall, 2010), these are the only ones under analysis in this study. The reason for choosing to concentrate only on the these three conceptualizations is that calculus textbooks often focus on area, anti-derivatives, and the Riemann integral during the exposition of and treatment of integrals (see, for example, Salas et al., 2006; Stewart, 2012; Thomas et al., 2009). Integrals in texts are often motivated by the study of irregular areas under the graphs of functions, approximated with finite Riemann sums, defined using the Riemann integral definition, and worked with using anti-derivatives. It is important to note that student conceptualizations are not necessarily *equivalent* to these ideas, but are rather *based* on them.

The way in which students cognitively possess these three conceptualizations is described through the lens of *symbolic forms* (Sherin, 2001). A symbolic form consists of a *symbol template* and a *conceptual schema*. The *symbol template* is the structure or arrangement of the symbols in the expression or equation, as in $\int_{[]}^{[]}[]d[]$ or $\int_{[]}[]d[]$, where each box "can be filled in with any expression" (Sherin, 2001, p. 490). The *conceptual schema*, on the other hand, "is the idea to be expressed" in those symbols (Sherin, 2001, p. 491). That is, it's the meaning that students see as being represented by the symbols in the definite integral structure. Three symbolic forms of the integral, called the *perimeter and area*, the *function matching*, and the *adding up pieces* forms, describe conceptualizations based on the area under a curve, anti-derivative, and Riemann sum conceptions, respectively. This paper does not analyze these symbolic forms in and of themselves, and the reader is referred to Jones (2013) for more detailed descriptions and an analysis of these cognitive structures. Rather, in this paper, these symbolic forms were used during the analysis for determining which conceptualization of the definite integral students were drawing on during a particular interview item. Therefore, it is necessary to briefly acquaint the reader with these three symbolic forms.

2.1.1. Perimeter and area

Each "box" in the definite integral symbol template is associated with part of the perimeter of a shape in the $(x-y)$ plane. The limits of integration become vertical lines for the "sides" and the integrand forms the "curvy top" of the shape. The symbol in the differential box, "*d*[]," dictates the variable that resides on the horizontal axis, which in turn forms the "bottom" of the shape. Within the perimeter and area symbolic form, the region of interest is held to be a fixed whole that is not divided up.

2.1.2. Function matching

The expression situated in the "integrand box" is interpreted as having come from some other, "original function." The original function has become the integrand via a derivative, and so the integral is thought of as a matching game, trying to get back to this original function. The symbol in the differential box, "*d*[]," dictates what the independent variable of the function is. The limits of integration are values that must be inserted into this original function to get a final numeric result.

2.1.3. Adding up pieces

The differential symbol, "*d*[]," is thought of as representing an "infinitesimally small" piece of the domain. Of these pieces, a single *representative piece* is cognitively selected to analyze the multiplicative relationship between the quantities represented by the integrand box and the differential box. The integral symbol itself implies that the resulting quantity of this multiplication is then added up over the infinitely many pieces, requiring an "infinite" summation. The limits of integration are the "starting" and "stopping" places of this infinite summation.

As an important part of this paper, the *adding up pieces* symbolic forms requires a bit more discussion here. In Jones (2013), it was demonstrated that, within this conceptualization, students tended to think of the Δx from the Riemann sum as having shrunk to an "infinitesimal" size, or size "zero," *before* the addition takes place. As a result, this conception is related to what Czarnocha, Dubinsky, Loch, Prabhu, and Vidakovic (2001) call the "method of indivisibles," which considers area under a curve to be the summation of straight lines that fill in the space underneath it. As an example of what I mean, consider Chris' explanation:

Chris: We want to find the area, so theoretically we could add up the value of a bunch of rectangles, and add them up. But we're going to constantly have little gaps... so we're going to be missing this area [points to small gaps in between the rectangles and the curve]. So we assume, by integrating, *we assume that dx is infinitesimally small* (Jones, 2013, p.126, emphasis added).

Chris seemed to believe that while the integral has its genesis in the idea of increasingly accurate approximations, the integral itself inherently involves infinitely-many, infinitesimally-small pieces, which is like the "method of indivisibles." In this way the adding up pieces conceptualization does not represent an *approximation* to the integral, but rather conceptually considers its exact value, since infinitesimal pieces do not "have little gaps." Therefore, in this paper, when students speak of "*dx*" I assume, as Chris states, that they mean that "*dx* is infinitesimally small." In turn, by "infinitesimally small," it seems that the students are referring to a width that has shrunk in size to zero, allowing for infinitely many pieces to fit side by side inside the domain prescribed by the limits of the integral.

However, there is an important distinction between "indivisibles" as described by Czarnocha et al. (2001) and how the students in Jones (2013) seemed to talk about the integral. That is, the students typically did *not* think of the rectangles with width *dx* as having "collapsed" in dimension to straight lines. Rather, even though a piece is "infinitesimally thin," it still retains a two-dimensional quality, more like a rectangle than a straight line. As an example, take Devon's description from that study:

Devon: I would imagine it as, you slice it [draws a thin rectangle] like very small pieces and each of them [draws an arrow from the bottom] is a *dx* [writes "*dx*"] (Jones, 2013, p. 126).

In imagining the definite integral, Devon clearly used the idea of "rectangles" as opposed to straight lines in depicting the "pieces." Like Chris, he had often associated *dx* with an infinitesimally thin width. Thus, as Devon described, this infinitesimally thin width still has some "substance" to it, despite having size "zero," meaning that the students still generally spoke of two-dimensional rectangles, rather than one-dimensional straight lines. This was also true for other types of differentials, like *dt* or *dA*.

A final important aspect of the adding up pieces conceptualization must be acknowledged here. Adding up pieces is *not* equivalent to the Riemann integral, since the students tend to think of the summation happening with an infinite number of infinitesimally small pieces. By contrast, the Riemann integral constructs a sequence of finite summations and then considers the limit of this sequence. Yet, while they are not equivalent, it is true that the adding up pieces conceptualization is rooted in the idea of the limit of Riemann sums.

2.2. Knowledge in pieces and context

The construct of symbolic forms came out of the general "knowledge in pieces" program (diSessa, 1988). An important conclusion that emerges from this view of knowledge is that it might be too simplistic to call something like the definite integral a single "concept" in someone's cognition (diSessa & Sherin, 1998; Hammer, Elby, Scherr, & Redish, 2005). The definite integral contains too many ideas in it, such as notions of curves, rectangles, functions, area, derivative rules, multiplication, summation, and so forth. Further, these ideas may or may not be interconnected in a students' mind, and a certain subset of them may be called up in one context but not in another. This idea challenges the classic notion of "misconceptions" in that it is possible for certain ideas to be useful in one context, but not in another (Smith, diSessa, & Roschelle et al., 1993/94). For example, it is possible that an interpretation of a definite integral through an "area under a curve" lens may suffice for making sense of a given mathematics problem, but fail to be sufficient for a different physics problem.

This idea is central to this paper. That is, according to the knowledge in pieces tradition, it may not be possible to objectively state that an area under a curve, anti-derivative, or Riemann sum conceptualization is inherently "good" or "bad." Context becomes the critical ingredient (diSessa, 2004). As a result, this study looks at these three conceptualizations in two different contexts, pure mathematics and applied physics, in order to make judgments on how useful or helpful each one is in those specific contexts. Consequently, the results of this paper should be viewed in this light – calling a conceptualization more or less productive is relative to the context in which the student was drawing on it to interpret a definite integral expression.

Since it is possible that not all aspects of knowledge regarding the definite integral are activated simultaneously any time a student encounters an integral, there is a tacit "choice" of which knowledge resources a student will draw on. This paper uses the notion of a student's *framing* of the situation to account, at least in part, for the activation of certain pieces of knowledge over others (see Hammer et al., 2005; MacLachlan & Reid, 1994). Framing is essentially how an individual interprets a given situation, which affects how that individual thinks or acts. A student's beliefs about what is being asked in a given problem or what kind of answers are expected in mathematics or physics work influence the type of knowledge regarding the integral that student might choose to activate and use. Thus, framing was used in the design of the study in order to intentionally create distinct mathematics and physics contexts in order to better detect how each conceptualization of the definite integral played out in each context.

2.3. Definition of "productive"

The purpose of this paper is to evaluate, in mathematics and physics contexts, the three symbolic forms that correspond to area under a curve, anti-derivatives, and limits of Riemann sums. These conceptualizations are evaluated for how helpful, useful, or beneficial they are for the task of *making sense* of definite integrals in mathematics and physics expressions and equations. Note that this implies the task is not about *computing* integral expressions, but is rather about *making sense* of them. I choose the phrase "making sense" instead of "understanding" deliberately, because there is a difference between being able to carefully articulate the precise mathematics underlying a particular concept (i.e. understanding) and having the general idea of how that concept applies to or relates to a given situation (i.e. making sense). As an example of this distinction, a person can use a concept like "standard deviation" from statistics to reason with and make sense of scientific data without necessarily having to mentally access all the careful details that go into the standard deviation formula. That is, if a doctor encountered a standard deviation related to the weights of newborns, they could make sense of that number as something along the lines of "the average amount that a given newborn likely deviates from the mean." However, in a technical mathematical sense, this is false. The standard deviation does not *exactly* compute the average deviation from the mean. Yet, it would be reasonable to say that this doctor is thinking of standard deviation in a way that it is helpful, useful, or beneficial for the task of *making sense* of newborn weights (see Triola, 2010). In a similar way, I investigate whether the three conceptualizations of the definite integral described above are helpful, useful, or beneficial for students in making sense of definite integral expressions. In this paper, I have chosen to use the word "productive" to capture this idea. In this way, the use of the word "productive" is supposed to suggest the *everyday use* of the word, and I am not attempting to create a theoretical construct around this word. Thus, in this paper, a conceptualization of the definite integral is defined to be "productive" if it is helpful in accomplishing the task of *making sense* of definite integrals.

In order to add some specificity to the use of the everyday word "productive," I require two characteristics in order to claim a conceptualization to be productive. First, the interpretation a student makes of an integral expression should be *based* on correct mathematical ideas represented by an integral, even if the student does not fully articulate all aspects of the mathematical ideas. This is in line with the "making sense" discussion above, in that it is possible to draw on mathematical knowledge in a way that is helpful without necessarily attending to all of the details that come along with the precise mathematical formulation. Second, the phrase *making sense of integrals* seems to imply that a conceptualization should not only be helpful in interpreting definite integral expressions, but that a student should *feel* satisfied with their explanation. Thus, even if a student makes a tentative statement that is in line with the mathematical principles involved in integrals, if that student never feels particularly satisfied with their explanation then the conceptualization cannot be considered fully productive for the task of *making sense* of integrals.

In Section 3.4, a description is provided of the methods used for determining whether a conceptualization in a given context was "productive." In this paper I only make a distinction between whether a conceptualization is considered *productive* in a given context or *less productive* in that context. I do not wish to assert that a conceptualization is ever completely "not productive" since its productiveness may be highly dependent on the particular interview context. Therefore, I do not want to give the impression that the conclusions in this paper regarding a conceptualization's productiveness represent a definitive judgment on that conceptualization for *all* contexts.

3. Methods

3.1. Student participants and data collection

To capture data of students discussing the integral, interviews were conducted with eight participants from a major West coast university in the United States. All eight students were interviewed in pairs, so that they could discuss their thinking with each other. The student pairs were each interviewed twice, with the second interview happening one week after the initial interview. The interviews were held in an empty classroom where only the researcher was present with the students, in order to present the tasks and ask follow-up questions. The students worked at the board on mathematics and physics problems and were expected to work together to answer the interview items to the satisfaction of both participants. The students were encouraged to verbally discuss their thinking with each other while they worked. Their written and spoken activities were videotaped and the researcher took notes. These comprise the primary sources of data for the study.

The students chosen for this study had two important characteristics. First, since a large portion of first-year calculus classes is made up of science and engineering students (Bressoud et al., 2013), it was natural to include students from these disciplines. Second, participants were desired to be experienced calculus students, so that any lack of productiveness of the three conceptualizations under consideration could not simply be attributed to a lack of developed mathematical knowledge (as in Christensen & Thompson, 2010). In order to recruit participants who met these two characteristics, students were all recruited toward the end of a calculus-based physics course at their university. Students were only invited to participate who had completed the first calculus course at their university and had either completed, or nearly completed, the second calculus course. Students were only selected who had received a grade of A or B in these courses, or had a score of five on the relevant AP exam. As a result, all of the students in this study had experience working with definite integrals in both mathematics and physics contexts. The pseudonyms given to the students are to help suggest the pairs they worked in: Adam and Alice, Becky and Bill, Clay and Christopher, and Devon and David.

3.2. Context-specific interviews

Since this study explores three conceptualizations of the definite integral in both mathematics and physics contexts, the interviews were set up to provide the needed distinction between the two contexts. The first interview was intentionally framed as a "mathematics" session, where the items were created to resemble problems, functions, and integrals found inside typical calculus curriculum (see Salas et al., 2006; Stewart, 2012; Thomas et al., 2009). The second interview, by contrast, was purposefully framed as a "physics" session, where the items were created to resemble problems and equations found in physics or engineering curriculum (see Hibbeler, 2012; Pytel & Kiusalaas, 2010; Serway & Jewett, 2008; Wilson et al., 2010). A listing of the interview items used for this study is found in Appendix A. The items are listed in the same order in which they were asked of the students.

3.3. Coding the data

As a preliminary step to analyzing the three symbolic forms of the definite integral that are evaluated in this paper, the interview data was broken into "episodes." An episode was defined as an entire segment of student work, whether brief or lengthy. An episode might consist of one student making a self-contained explanation by his- or herself, or of both students having a continuously flowing interchange between them. After the data was chunked into episodes, each episode was coded into the following four categories: (a) perimeter and area, (b) function matching, (c) adding up pieces, and (d) none of these. An episode was coded into a category based on whether there was evidence from the discussion of one or both students cognitively drawing on that particular symbolic form. Note that an episode was coded into more than one category if the students showed signs of drawing on more than one notion of the integral during that episode. If a particular episode showed no evidence of any of these three symbolic forms, it was coded into "none of these." Note that the episodes in "none of these" typically dealt with narrow, specific aspects of the integral symbol structure, such as a "0" in the lower limit with a "2" in front of the integral depicting symmetry. Next, each episode was also placed into the overarching classifications of "pure mathematics" or "applied physics" depending on the interview during which the episode happened. By doing so, each episode was now tagged with two codes: one regarding the specific symbolic form(s) being drawn on by the student(s) in the episode, and another regarding whether the episode happened during the mathematics-framed interview or the physics-framed interview.

3.4. Analysis of productiveness

Once the episodes were sorted into this two-tiered coding structure, the work of analysis on each symbolic form's "productiveness" began. Note that the analysis was only conducted on episodes within the categories representing the three symbolic forms used for this paper and not on episodes in the "none of these" category. (For an explanation of the choice of these three symbolic forms, see Section 2.1.)

As stated earlier, the use of the word "productive" in this study is meant to suggest the everyday meaning of being useful, helpful, or beneficial for the task of *making sense* of definite integral expressions and equations. For this study, a conceptualization is considered productive based on two characteristics. First, if a student makes an interpretation based off of one of the symbolic forms of the integral, the interpretation needs to be *based on* the mathematical ideas inherent to the definite integral. However, as discussed earlier, the phrase *making sense* implies that the students need not attend to *all* aspects of the rigorous mathematical structure in order for the conceptualization to be useful in interpreting and explaining a definite integral expression. A conceptualization within an episode was analyzed along this dimension based on whether the students' descriptions and explanations of the definite integral in question led to reasonable claims and conclusions based on mathematical principles related to the integral. In the case of physics-based integrals, the explanations also needed to be in harmony with the scientific relationships between the quantities represented in the integral expression or equation.

Second, the phrase *making sense* seems to imply that the students themselves should be satisfied by their own explanations based off of the conceptualization they are drawing on. Since this second characteristic is more difficult to deduce, several complimentary dimensions were considered together to make a case for whether a student was satisfied by the explanation they provided. Note that no single dimension alone was used to determine whether a student was unsatisfied with a particular interpretation of the integral. Rather, several dimensions together had to point to a student being unsatisfied before labeling the student as such. In particular, several "semiotic resources" were used for determining whether a student was satisfied or not (see Arzarello, Paola, Robutti, & Sabena, 2009; Radford, 2003). Semiotic resources used in the analysis include gestures, glances, hand signals, considerable pauses, fragmented speech, and tone of voice. In addition, more direct forms of communication were also used as evidence of an unsatisfied student, such as hedging words or direct assertions of confusion.

It is important to note that in determining whether a given conceptualization was productive, the benefit of the doubt was given to a student being *satisfied* with their response. That is, a student was only deemed *unsatisfied* if enough evidence could be compiled that pointed to that conclusion. In this way, the null assumption was that a given conceptualization of the definite integral was productive and the burden of proof was on amassing evidence to conclude that the student was struggling to make sense of the integral expression or equation.

To alleviate the potential subjectivity of what combinations of semiotic resources and other forms of communication allow for the conclusion that a student was not satisfied, two things have been done. First, 19 episodes (one-third of the total) were given to a fellow mathematics education researcher with a strong research background to perform basic interrater reliability. Out of the 19 episodes, our initial ratings of productiveness agreed for 17 out of the 19 episodes (89.5% agreement). After a clarifying discussion regarding the context of one particular episode, we agreed on 18 out of the 19 episodes (94.7% agreement). Second, extended excerpts of several students' explanations are provided in the results section, so that the readers may make judgments for themselves regarding whether the student has been properly categorized as satisfied or unsatisfied with their response.

3.5. Compiling results and selecting example episodes

After chunking the data into episodes, coding the episodes, and analyzing the three symbolic forms within each episode, I created a master list of the three conceptualizations' productiveness. To do so, I created a table that listed each interview item, each symbolic form drawn on by students within that particular interview item, and whether each symbolic form was productive or not for that particular student or student pair during that item. This table served as the basis for making overall judgments of each conceptualization in terms of how productive it was in each context. This table is displayed in Section 4.3.

After determining each symbolic form's productiveness in both contexts, I made selections of specific episodes to use as examples in the results section of this paper. In order to best compare and contrast the three symbolic forms amongst themselves, I searched for episodes in two ways. First, I looked for episodes from the same interview item that would clearly contain all three symbolic forms, so that their productiveness could be easily compared using the exact same integral expression or equation. Second, I looked for episodes that highlighted some of the typical ways students drew on each of the conceptualizations. That is, the examples used are not unique in the way in which the symbolic form was productive or not, but rather capture some of the trends across the student data.

3.6. Limitations

I should acknowledge that analyzing student responses, spoken or written, only approximates an analysis of their actual cognitive structures, meaning that the students may have held more subtle understandings of the integral than this analysis can give them credit for. It is possible that these subtle differences make an impact on the productiveness of a particular conceptualization. Another important note regarding the student data must be made here. To properly analyze each conceptualization of the integral for its productiveness, each one had to be artificially isolated. Hence it may give the false impression that a student holds or draws on only a single conceptualization of the integral at a time. Yet this is merely a consequence of the need to isolate each conception of the integral in order to analyze it.

4. Results

I now discuss the productiveness of the *function matching*, *perimeter and area*, and *adding up pieces* symbolic forms of the definite integral. Rather than begin by displaying the results of the entire body of data first, I begin instead by presenting example episodes of each symbolic form in the mathematics and physics contexts. I do this in order to compare and contrast the three conceptualizations as I build an argument for why each one may be considered productive or less productive for a given context. After an in-depth discussion of these specific examples, I then shift focus to look at all of the conceptualizations more broadly across all of the interview items, to identify trends in their context-specific productiveness. In particular, in Section 4.3, I summarize the results of the whole body of data in order to make more general conclusions regarding the productiveness of each conceptualization in the mathematics and physics contexts.

I first discuss the three symbolic forms in the mathematics context and then subsequently move the discussion to the physics context. In brief, while all of the symbolic forms showed usefulness for decontextualized integrals, there was a significant difference in how productive certain understandings of the integral were when considering contextualized integrals. Here, I am using "contextualized" and "decontextualized" in their common meanings and am not referring to theories that use these words for specific purposes. For this paper, an integral is said to be decontextualized if it is devoid of any association to physical phenomena or any relation to a problem larger than the integral itself. For example, if the expression $\int_0^1 x^3 dx$ is simply given to a student, it is decontextualized, since it stands by itself. By contrast, a contextualized integral is one that is either connected to a larger problem or that has relations to real-world quantities. For example, the physics integral $mass = \int_R \rho dV$ is considered contextualized since the symbols deal with the physical quantities mass, density, and volume, and thus the integral does not stand by itself. These quantities provide an additional layer of meaning to the integral and the placement of the quantities in the integral are governed by physics properties.

4.1. The three conceptualizations in the mathematics context

In looking at the pure mathematics context, all three of these conceptualizations appear to be productive, so long as the integral in question was a decontextualized expression. This should not be surprising for the simple reason that areas under curves, anti-derivatives, and the limits of Riemann sums comprise different ways to approach the basic definitions and uses of the integral in mathematics contexts (see Salas et al., 2006; Stewart, 2012; Thomas et al., 2009) and the three symbolic forms discussed here are based off of these mathematical ideas. Consider the following interview excerpts taken from students' explanations regarding the *meaning* of the integral $\int_1^2 (2/x^3 - x^2)dx$.

David: In an integration the *dx* is always essential, because it shows that this entire thing [waves his hand over the integrand, "$2/x^3 - x^2$"] is a derivative of *x*. . . The fact that this entire thing is sitting right next to each other, and *dx* outside, means that basically this entire function [sweeps his hand over "$2/x^3 - x^2$"] is the derivative of an original function.

Chris: Like, if we have two curves [draws a curve and a second curve below it], instead of having a single integral to solve this total area all at once, we're finding the integral of the top one [spreads his hands from the upper curve to the horizontal axis] and then we're subtracting this area [outlines shape from the lower curve to the horizontal axis; similar to Fig. 1].

Bill: Well, if we take a small area, like say this [draws a thin vertical rectangle between the upper and lower curves], and this distance here is *dx* [underlines the thin bottom side of the rectangle]. . . We're just splitting into lots of rectangles that way. . . If we take this rectangle, and put it up here [draws a "zoomed in" version of the thin rectangle; similar to Fig. 1], this would be *dx* right here [traces the width of the rectangle]. And this would be, the difference here, would be the length, the $f(x)$ minus $g(x)$. . . [The integral is] the rectangles, the sum of all the rectangles.

Each student made quite a different interpretation of the same integral, yet each explanation is based on the mathematical notions of area, anti-derivatives, and the summation of multiplicatively-constructed quantities, which are all ways in which definite integrals are treated in typical texts (see Salas et al., 2006; Stewart, 2012; Thomas et al., 2009). Furthermore, each student appeared satisfied with their response. Thinking of an "original function" that became $2/x^3 - x^2$ through a derivative allowed David to explain why it makes sense to "reverse" derivative rules in order to calculate definite integrals. By visualizing shapes in the plane, Chris thought of removing the "lower" portion from the overall shape to get the area of interest. Bill used a representative rectangle to explain the relationship between the integrand and the differential, and how the integral adds up the quantities represented by those rectangles. As already discussed in Section 2.1, I claim that Bill's use of "*dx*" for the width of the rectangle implies that he is thinking of it as being "infinitesimally small." By doing so, Bill is not thinking of an *approximation* of the area, as with a finite Riemann sum, but is thinking of an infinite number of infinitely thin rectangles filling in the area *exactly*. While this makes the adding up pieces not equivalent to the Riemann integral, it does seem to allow Bill to *make sense* of why the integral expression calculates the area in between the curves.

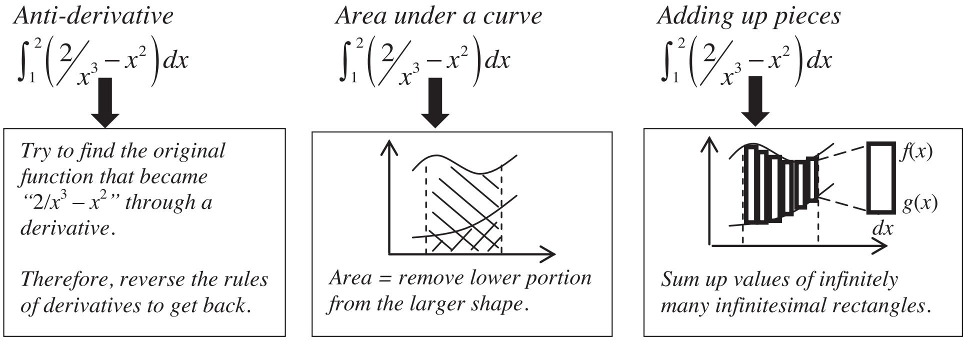

Fig. 1. All three conceptualizations are productive for decontextualized integrals.

Note that for the purposes of *making sense* of the integral, it does not matter that Chris and Bill did not realize that $2/x^3$ is sometimes beneath x^2 in the interval [1,2]. This oversight was not dependent on the conceptualization of the integral they were each working with. Both students showed evidence later in the interviews of understanding "negative area" and the overall ideas contained in their responses were still valid and made sense to the them.

For each of these students, follow-up questions such as, "What does the '*dx*' mean?" and "What do the 1 and the 2 mean?" were asked to investigate whether each student seemed confident in terms of how they were making sense of this integral. In addition to providing statements based on mathematical meanings of the definite integral, each student continued to be satisfied with the perspective they were drawing on, namely the function matching, perimeter and area, and adding up pieces notions respectively. Thus, it seems that students might approach decontextualized integrals through whichever of these interpretations they wish, and each provides the students with a mathematically reasonable explanation that makes sense to them. Thus, all three symbolic forms can be considered *productive* in this context (Fig. 1).

From this basis, it would seem that none of these conceptualizations is any more or less productive than any other, in terms of making sense of what an integral means. However, the issue is that many important problems involving integrals do not come in the form of interpreting decontextualized integral expressions that are simply given to students. This leads the discussion to the next context, in which the students looked at integrals from physics and engineering.

4.2. *The three conceptualizations in the physics context*

To see how these three conceptualizations of the integral played out in understanding applied physics and engineering expressions and formulas, I first consider how they affected student understanding during interview item Physics2 (see Appendix A). In this task, the students were asked to interpret the meaning of the integral $\int_0^{600} R dt$, where R represents the varying revolutions-per-minute of a motor that runs for a 10-hour period. I begin with the function matching symbolic form, which interprets the integral as coming from the derivative of some "original function." In particular, Adam and Alice relied on this notion during both interviews. While working on this task, they stated that the integral calculates the total number of revolutions over the 10-hour period, so I asked them to explain why.

Alice: Like, let's just say you had a velocity function, which equals meters per second. You know when you take the integral of that, it gives the position, it will just be meters. So you just apply the same thing to that [points to the integral of *R*].

Alice drew a comparison to an integral of the velocity function over time in order to justify why the integral resulted in a measure of revolutions. I pushed her and Adam to discuss the relationship further.

Interviewer: Let me go back... how is it that that formula right there [the integral of *R*] ends up just giving you revolutions?
Alice: Well, um, you said that this denoted revolutions per minute, just like velocity denotes meters per second. And so by applying what we have learned before about how this equation [points to *m*/*s*] goes to that [points to *m*], we know that it would just go to revolutions.
Interviewer: And why would it do that? Why does velocity, meters per second, why does that go to just meters? And why does revolutions per time go to just revolutions?

[pause of considerable length]

Alice: [quietly, hesitantly] Just like, the integral of the function, so from that it would cancel out...

...

Adam: It's just... shoot. Like velocity equals, uh, like... So x equals [writes "x="]... shoot. [Writes "$v = dx/ds$ and $a = dx^2/ds^2$." Note that Adam is using "s" as the time-variable.] So it's kind of just like, just the next step. Acceleration would be ds squared and then you integrate that out and... I'm not sure why, but just one less ds here. And you use the next step for position. This, the velocity represents, like, revolutions per minute [points to R inside integral], so that ds would just kind of come out.

Interviewer: So what would be your best guess as to why to that happens? So why it goes from the ds squared to just ds?

[pause]...

Adam: I would say that acceleration is the derivative of velocity and the derivative of this [points to v] would just be... the... [considerable pause]. Take that back. So...

...

Adam: So, I guess the general derivative would be just like dx, derivative of x [writes "derivative $x = dx$"], uh whatever. So let me take the derivative of that, you get another one of these, or another dx/ds and you just multiply it. And that's the best I got.

Interviewer: Anything you'd add to that, Alice?

Alice: Um, each time you take the derivative, the equation will be changing. So if you have like x^3, take the derivative, it would be $3x^2$, so the power changes. So like for velocity and acceleration, let's just say that equals x^3 meters per second squared. And for velocity, $3x^2$ meters per second. So you can see from, like here, once you take the derivative this power changes [points to the "squared" exponent on the "m/s^2"] because it's no longer in the same formula [points to x^3 and x^2].

Here Adam and Alice clearly attempted to understand this integral through the anti-derivative-based function matching symbolic form. As outlined in Section 2.1, they both conceived of the integrand in question, R, as being the derivative of something else, as evidenced by the analogy they used that velocity is the result of the derivative of position. Second, their work followed the pattern of trying to identify what the original function was that became R through a derivative. Most of their thinking had to do with what happens under differentiation and trying to work "backwards" from derivatives to understand why the resulting units should be only in terms of revolutions, not revolutions per minute. Adam tried to bring in the idea that each anti-derivative would result in "one less ds" (or dt) and Alice tried to use what happens to constant exponents under differentiation, which decrease by one, to try to explain why m/s^2 becomes m/s and then m under repeated integrations.

However, there was a considerable lack of confidence displayed by Alice and Adam during this episode. There were frequent long pauses, many false starts, quiet tones of voice, nervous tapping of the markers, sentence fragments, backtracking, and statements such as, "I'm not sure why" and "That's the best I got." It is clear that neither Alice nor Adam were confident that their answers were correct. In addition to that, the function matching game used by Adam and Alice in this vignette failed to provide them with an understanding that generally agrees with how integrals are thought of in physics and engineering. The function matching symbolic form did not provide Adam and Alice with the multiplicative relationship between R and dt in order to realize that a given revolutions per minute over a certain amount of time would yield the number of revolutions over that time period. Neither were they able to make sense of that phenomenon with velocity over time, as a way to explain why m/s becomes m under integration. Consequently, this conceptualization failed to help the students make sense of the integral or the underlying physical phenomenon. Overall, I conclude that the anti-derivative conception is *less productive* in this context.

Now consider the area conceptualization applied to this same problem. Clay and Chris drew on this meaning as they initially approached the integral $\int_0^{600} R dt$. Typical of perimeter and area thinking, Chris drew a set of axes, made a squiggly graph in the plane, marked off $t = 0$ and $t = 600$ with vertical lines, and shaded in the region he had created (Fig. 2).

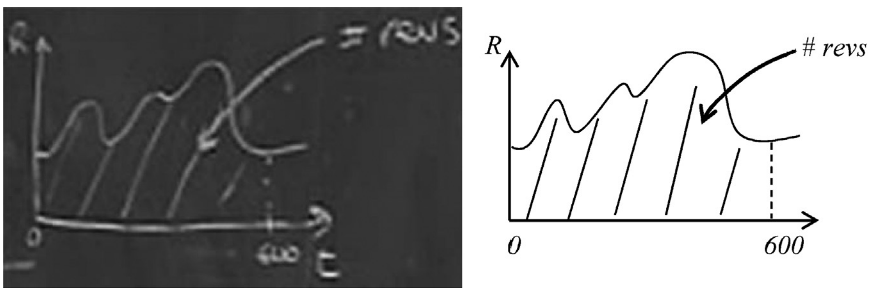

Fig. 2. "Area under a curve" drawing (with reproduction) by Chris.

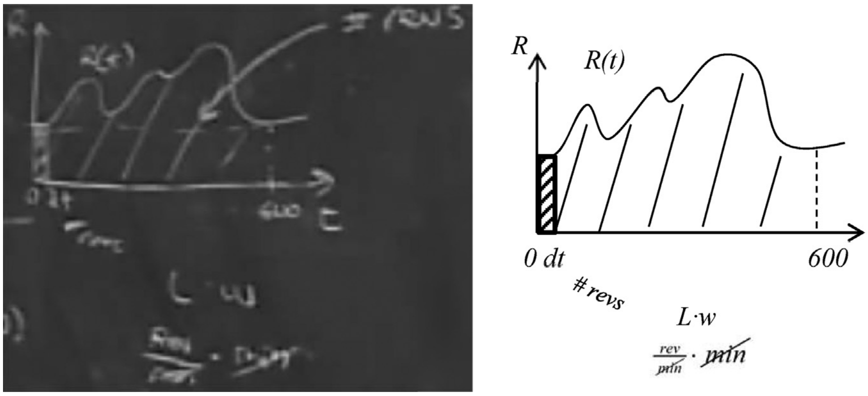

Fig. 3. "Adding up pieces" boardwork (with reproduction) by Clay and Chris.

Chris: We're going to be integrating from 0 to 600. And finding this area [waves hand over the enclosed shape] and this area is essentially going to be our... um... [pause]... number of [hesitantly] revolutions... It should be... Right?... This area here, below the *R*, should just be revolutions. Number of revolutions. [Quietly to Clay] Right?

...

Interviewer: In this problem, why do the minutes end up canceling out?
Chris: So the integral is essentially, is... [quietly] how do I write this? Hmm...

...

Chris: So we would just be finding the number of revolutions. [Quietly] I don't know if that's right.

Chris was quite comfortable with the familiar picture of an area under a curve. He created it often during his interviews, marking off vertical lines and shading in the resulting area. However, once this picture was created, Chris seemed confused. His manner of speech indicates a lack of confidence with his answer. There were pauses, quiet speech, and the hedging statement, "I don't know if that's right." Even though he correctly identified what the integral calculated, he struggled to justify why the shaded region of the plane should represent the quantity "number of revolutions." There was nothing in the picture he had created that showed why a function of revolutions per minute should somehow give the result "revolutions" under integration. He seemed unsatisfied with his explanation, given that he ended it by stating he didn't "know if that's right." While there is not anything *wrong* with the mathematics or physics represented by the picture he drew and the area he created, it did not help Chris make sense of *why* the integral computed what it was supposed to.

Bill was similarly confused while working on this same interview item. He also was drawing on the perimeter and area notion, evidenced by the typical drawing of a squiggly graph and a shading in of the resulting region (much like Fig. 2). However, after drawing this figure, he also seemed confused about what the resulting "area" should mean.

Bill: So if we take the integral of this [points to the region], what would that signify? Like, what does this area represent?

It is clear from his question that the area under the curve conception alone does not seem adequate for determining what quantity the integral calculates. These two examples suggest that the area conceptualization was *less productive* for this context.

By contrast, the adding up pieces symbolic form proved productive for understanding this same integral. After getting stuck during their initial explanation of $\int_0^{600} R dt$ as an area under a curve, Clay and Chris demonstrated a shift toward adding up pieces thinking. This thinking allowed them to move past their difficulty and successfully describe the meaning of the integral and how it "works."

Clay: We're solving with respect to the, so like each of these would be number of revolutions for that minute [draws a thin rectangle; see Fig. 3]. So this would be just *dt* [writes "*dt*" along bottom of rectangle], it would be the number of revolutions for this *dt* [writes "# revs" next to the rectangle]. And when you integrate this, it adds up each component.
Interviewer: So why does that little rectangle, why does that end up counting up the number of revolutions for that piece?
Chris: Yeah, so like, [to Clay] good call. We have *R*, which is revolutions per minute. So that means this side is revolutions per minute [traces finger down the height of the rectangle]. So we're going to multiply the length times width

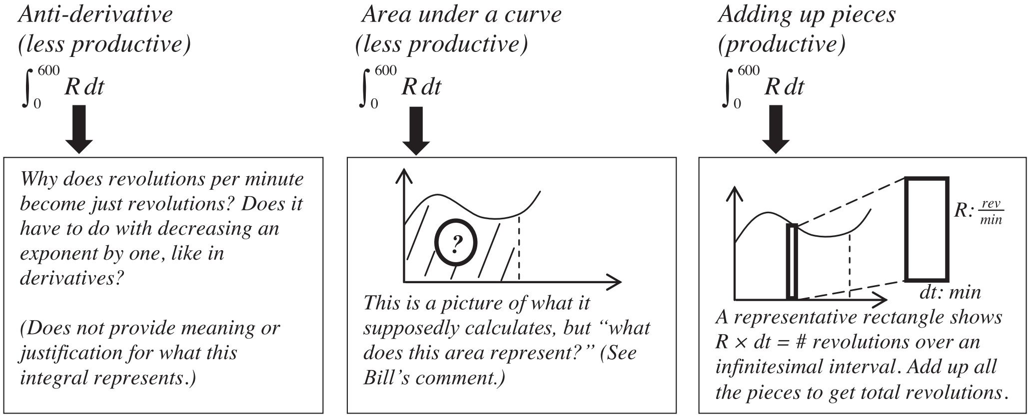

Fig. 4. The adding up pieces conceptualization proved productive for this context.

> [writes "*L·w*" under the graph]. Length would be revolutions per minute [writes "*rev/min*"]. And then our width is *dt* and that's just saying *dt* is essentially just going to be minutes, the time value [writes "*rev/min·min*"]. So the area of that rectangle would just be revolutions [cancels out both "*min*"].

By switching to this thinking, they were able to utilize a *representative rectangle*, characteristic of the adding up pieces symbolic form, to describe how *R* and *dt* interacted via multiplication to produce $\frac{rev}{\cancel{min}} \cdot \cancel{min} = revolutions$. Then, as Clay stated, all of the "numbers of revolutions" over each piece are "added up" to capture the total number of revolutions the motor experienced during the entire 10-hour period. As discussed before, the students throughout the interviews used the differential, such as "*dt*," as part of the rectangle to imply that they conceived of the rectangle as having an infinitesimal width. This conceptually allows for infinitely many slices to be placed under the curve, which permits the consideration of calculating an exact value, as opposed to an approximation.

Thus, the adding up pieces conception, together with its signature feature of a representative rectangle, was *productive* for these students, even where the anti-derivative and area under a curve interpretations failed to be (see Fig. 4). Ironically this conceptualization gave rise to a meaning for what the area represented, even when the area conception itself was unable to.

In a second example, I look at the three notions of the integral in relation to interview item Physics3. This item asked the students to explain why the integral expression in the equation $F = \int_S PdA$ calculates the total force on a surface, *S*. Clay and Chris initially began to describe this integral through the function matching lens that an anti-derivative with respect to the variable *A* was needed to calculate this integral. However, this "*A* as the variable" interpretation of *P* produced a problematic response.

Clay: Like, will that [points to *P*] be a function of *dA*?
Chris: That means *P* has to be a function of, uh, with respect to *A*, I guess. Because... yeah, so we would have to be given, in order to solve this with actual values, we would need to be given *P*, pressure, as a function of area.

...

Chris: [quietly to Clay] How would you say that? [Long pause.]

The idea that the differential, *dA*, dictates what the independent variable of the integrand is steered the students toward assuming that *P* had to be a function of *A*. In this case, it appears the students were thinking of *P* as a function of *A* algebraically, which would look something like $P(A) = A^2 + \cos(A)$, for example. However, pressure is typically used as a function of *location* as opposed to the overall surface area, as in $P(x,y) = x^2 + \cos(y)$. While it is possible to consider pressure as a function in this way, it is not the usual way pressure is discussed in physics and engineering texts (see Hibbeler, 2012; Serway & Jewett, 2008). In the language of Von Korff and Rebello (2014), Chris and Clay are thinking of a "change" infinitesimal, wherein "*dA*" represents a change in area, as opposed to an "amount" infinitesimal, wherein "*dA*" represents a small amount of area. Chris and Clay seem to be reducing the relationship of pressure and area to a symbolic, algebraic one, instead of one of multiplication between two quantities. In doing so, it appears that these two students became confused regarding the meaning of the definite integral. Thus, the anti-derivative conception was *less productive* for these students, since it led to incorrect assumptions about the integrand and the differential. I note here that after getting stuck at this point, Clay and Chris again exhibited a shift toward *adding up pieces*, which greatly improved their understanding of this integral. This shift is described later in this section.

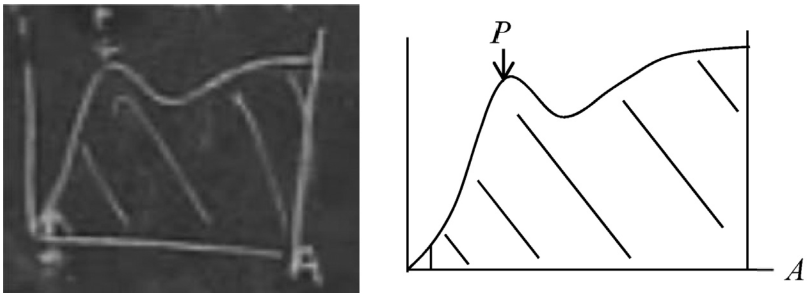

Fig. 5. "Area under a curve" drawing (with reproduction) by Bill.

Next, consider the area under a curve conceptualization applied to this same integral equation through Bill's work. This particular example was reported previously in Jones (2013), but a more fleshed out analysis of the episode is given here. Also, in that paper, this example was provided in isolation, simply as a way of motivating the possible utility of the adding up pieces symbolic form. By now combining this example with the example from Clay and Chris, a richer analysis is achieved and a much clearer picture can be seen of how the adding up pieces symbolic form differs in its explanatory power for students from the function matching and perimeter and area symbolic forms.

Bill began the discussion of this interview item by drawing a set of axes, making a squiggly graph, labeling the graph *P*, labeling the horizontal axis *A*, marking off vertical lines for the left and right sides of a bounded region, and shading in the enclosed shape (see Fig. 5). However, Bill seemed unsatisfied as to why the area of the enclosed shape should be "force."

Bill: So, if we took the integral of that, it would be, it would be all this... [shades in the region underneath the *P*-graph]. And that would be the total force it would exert... [pause]... I feel like I skipped a step.

...

Interviewer: So conceptually, what does *dA* mean?

Bill: The change in area [points to "*A*-axis"]... It's hard to say... But I think if you took the, if you measured the pressure and there's that much area [puts chalk on a point on the "*A*-axis"], and you took the pressure of some other lesser area [puts chalk on another point on the "*A*-axis"], and you subtracted, that would be what *dA* is, the change in area. [Pause]... If I said that right [laughs].

Later he added the following two statements:

Bill: I think what I said before was kind of wrong.

Bill: I guess my real problem I'm trying to figure out is how to relate what *dA* exactly is.

Bill's reliance on the area under a curve conception motivated him to create a one-dimensional curve in the plane to model the pressure over the surface area. He was led into inappropriate notions of *dA*, since he assumed the variable *A* should be represented along the horizontal axis, typical of perimeter and area thinking. This idea gave rise to "changes in area" even though the problem stipulated a single, fixed surface over which a force was exerted (see Von Korff & Rebello, 2014). Bill, himself, recognized this, saying that he didn't really know "how to relate what *dA* exactly is." His hesitancy is also shown by the statements, "I feel like I skipped a step" and "It's hard to say," as well as the fact that he later declared that his explanation was likely "wrong." The area under a curve conceptualization did not help him in understanding what the "area under the curve" actually meant. Consequently, this interpretation of the integral is *less productive* for understanding this equation.

Like Clay and Chris, Bill later shifted his thinking to adding up pieces, which enabled him to satisfactorily interpret the meaning of this integral. These two cases of the shift toward adding up pieces are now explored.

I return here to Clay and Chris, who had initially made incorrect assumptions about the nature of the integrand, *P*. After their previous discussion, they decided to draw a picture and used a large rectangle to represent the surface, *S*. Chris then drew a small *representative square* inside of the rectangle (see Fig. 6).

Chris: So *dA*, which is this [points to the small square], is equal to *dx* times *dy* [writes $dA = dx \cdot dy$]. This is *dx* and this is *dy* [points to the sides of the small square].

...

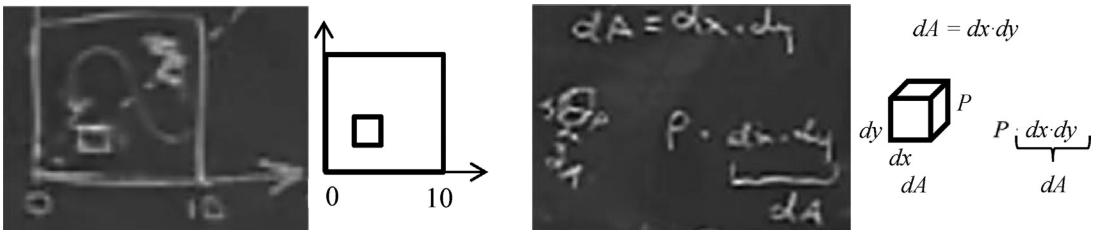

Fig. 6. Representative square and cube (with reproductions) by Chris and Clay.

Chris: So, force equals pressure times area. We have the area [points to the small square] and we have the pressure function [makes "incoming" motion toward the square].

Clay: And by making it an integral you have all the small bits of area times the pressure at that location, and you have the total force because of that.

Chris: Yeah, so, like as we were talking earlier, instead of having a rectangle [draws a rectangle], this was *dt* and this was *R* [a reference to interview item Physics2; see Fig. 3], now we have, essentially, a cube [draws a cube, see Fig. 6], where this is *dx*, *dy*, and this is pressure [labels the sides of the cube]. Or just *dx* and *dy* being *dA*.

. . .

Chris: And then we have pressure times area [points to $P{\cdot}dx{\cdot}dy$], so we're actually finding, so the volume of this [points to the cube] essentially is force. Since we're finding the integral of these infinitesimally small cubes, which consist of pressure and then the *dx* and *dy* which would be *dA*. So, we're integrating over these infinitesimally small volumes, which each one composes of a force, so we're integrating force, and adding up all the infinitesimally small pieces of force, to find the total force.

Unlike the unproductive ideas contained in the discussion of *P* being a function of *A*, thinking of breaking the surface down into small squares enabled Chris and Clay to break past their difficulty and understand how this integral works. They drew on a representative square, which they then turned into a representative cube. By "zooming in" on this cube they could analyze the properties of the integral. The bottom of the cube, given by *dx* and *dy* was actually the differential *dA*. This differential quantity, *A*, was multiplied to the integrand, *P*, to produce a "small bit" of the resultant quantity, force, which comprised the volume of that small cube. The "small pieces of force" contained in each cube were then added up in order to "find the total force" on the entire surface. Once again, their use of differentials depicts their thinking of the differential *dA*, or the related *dx* and *dy*, as infinitesimally small. Chris even states this explicitly, saying that the "infinitesimally small volumes" represented "infinitesimally small pieces of force," which could be accumulated to find the exact amount of force (as opposed to an approximation). While, again, this conception is not equivalent to the Riemann integral, which constructs a sequence of values from finite Riemann sums and then takes the limit of the sequence, it is consistent with how physicists and engineers often think about and use definite integrals (see Hibbeler, 2012; Pytel & Kiusalaas, 2010; Serway & Jewett, 2008; Wilson et al., 2010). Furthermore, it provided the students with a conceptual framework for *making sense* of this definite integral equation.

The adding up pieces conceptualization allowed Clay and Chris to fully deconstruct the integral into its component pieces and put them back together in a way that explained how the integral computes force. Thus, this notion is *productive* for this applied context (see Fig. 8). The same result held true for Bill, once he switched from the area under a curve conceptualization to adding up pieces. After he and Becky failed to make progress for a while, I asked them to think about the table I was sitting at as the surface, *S*, and to tell me if the integral would apply to that situation.

Bill: I believe that, uh, I'm just trying to relate this to rectangles. If we just took the area of this piece of the rectangle here, this part of the table, and found the total force exerted on that, you would get some kind of estimate.

. . .

Bill: [Draws a rectangle on the board to represent the table.] Let's just say this is *dA* [draws a small strip at one end of the rectangle, see Fig. 7]. This whole thing is *dA*, this whole area [again references the small strip]. And you have pressure pushing on that, on all that area [writes "*P*"]. So you can multiply *P* times *dA* and you get the total force pushed, exerted on that part of the table [writes "*F*," see Fig. 7].

. . .

Bill: Yeah, if you make that area smaller and smaller and smaller and then add up those infinite, those really small areas on the whole table, you get the total force.

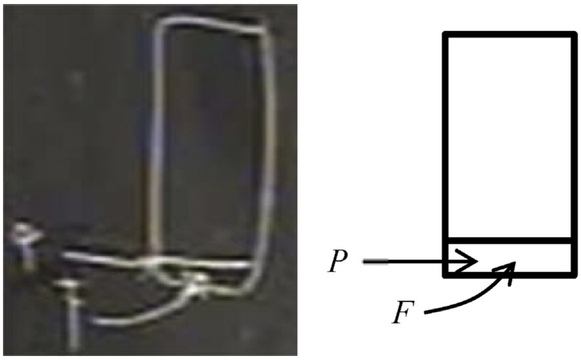

Fig. 7. “Adding up pieces” boardwork (with reproduction) by Bill.

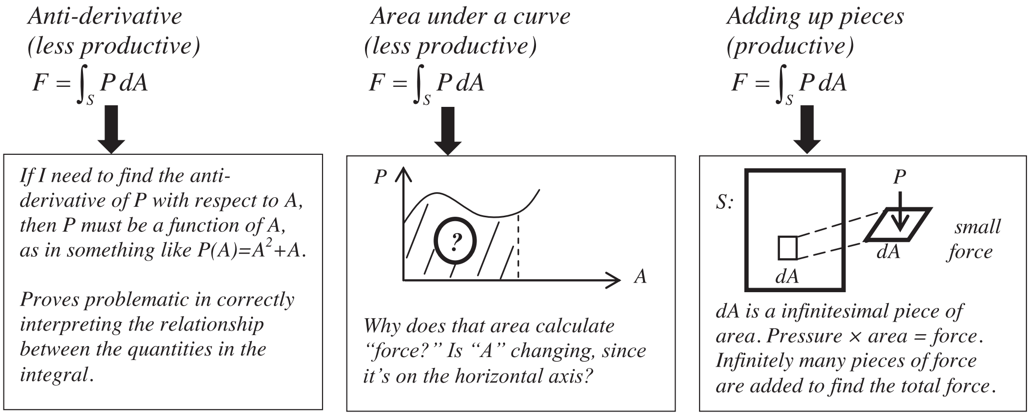

Fig. 8. Adding up pieces proved highly productive for applied integrals.

Once Bill began to conceptualize the table as being divided into small pieces, each one being a *dA*, he was able to relate pressure and area over each piece. These combined, through multiplication, to give the force “exerted on that part of the table.” Then the “infinite” number of small bits of forces were all added up to give the “total force” on the entire surface. Again we see the productiveness of the adding up pieces conceptualization for this problem, where both the anti-derivative and area under a curve notions proved less effective. Note that Bill’s explanation did not require any advanced understanding of multivariate calculus. He was able to simply divide the surface into smaller pieces and visualize an addition of force over all those pieces. Thus, this conceptualization is *productive* in this context (see Fig. 8).

As a final example, the adding up pieces symbolic form enabled the students to make sense of a *variety* of applied integrals by helping them attend to the relationship of the quantities represented by the integrand and the differential. Consider Devon’s explanation, while working with interview item Physics1, about why an integral is appropriate for calculating the mass of a box. Devon created the integral $\int_V \rho(r)dV$, so I asked him to explain what it meant.

Devon: *dV* would be every small piece of the volume... It would just be a little box, like a bunch of little boxes [draws a row of small cubes inside the larger box; see Fig. 9]. And each of them has a rho [ρ], has its own density at that particular piece.

...

Devon: You find out the mass of every small piece, and then you just add them up together [circles hand around the *V* subscript on the integral].

...

Devon: Each box has its density and then you calculate the mass of each box. And then you add them up.

Devon envisioned the box being divided into small pieces and equated the differential, *dV*, to each small piece. He then related the integrand, ρ (density), to each *dV*, which multiply together to give the mass of that tiny section of the box. The integral itself indicates that all of the little pieces of mass should be added up from throughout the entire box to find the

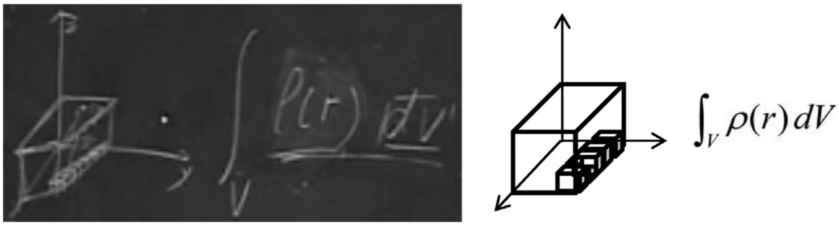

Fig. 9. "Adding up pieces" drawing (with reproduction) by Devon.

Table 1
Adding up pieces (AUP), function matching (FM), and perimeter and area (P&A) per interview item (in alphabetical order). Each is shown as productive (P) or less productive (LP).

	Alice/Adam	Bill/Becky	Christopher/Clay	David/Devon
M1	AUP – P P&A – P	AUP – P FM – LP P&A – P	AUP – P P&A – P	AUP – P FM – P P&A – P
M2	FM – P P&A – LP	AUP – P FM – P P&A – P	AUP – P FM – P P&A – P	FM – P P&A – P
M3	FM – P P&A – LP	FM – P P&A – LP	AUP – LP	AUP – P FM – LP P&A – P
M4	P&A – P	AUP – P	AUP – P P&A – P	FM – P
P1	AUP – P	AUP – LP	AUP – LP	AUP – P
P2	FM – LP	AUP – P FM – LP P&A – LP	AUP – P P&A – LP	AUP – P FM – P
P3	AUP – P	AUP – P P&A – LP	AUP – P FM – LP	AUP – P
P4	–	AUP – P FM – LP P&A – LP	AUP – P	AUP – P

overall mass of the object. Once again, the use of *dV* for each piece suggests that Devon conceptualized them as infinitesimally small, providing an exact value for the box's overall mass. Thus, for this context, the adding up pieces conceptualization was once again *productive*, allowing this student to understand, analyze, and make sense of the integral expression.

4.3. Overall productiveness of the three conceptualizations

In this section, I shift focus to discuss the results from the entire corpus of data. By doing so, I make conclusions regarding the helpfulness or usefulness of these conceptualizations more broadly in the mathematics and physics context. Table 1 shows the symbolic forms drawn on by each student pair, per interview item (see Appendix A). For each instance, labels are added for whether that conceptualization was productive (P) or less productive (LP) for that particular interview item context.

After creating this table, I recorded the number of instances in which each conceptualization was rated as productive or less productive. Furthermore, this count was conducted for the mathematics context and the physics context separately, so that the overall productiveness of each conceptualization can more clearly be seen in each context. These counts are represented in Table 2.

Table 2
The overall productiveness of each conceptualization in math and physics contexts.

	Productive	Less productive
Math context		
Function matching	8	2
Perimeter and area	10	3
Adding up pieces	9	1
Physics context		
Function matching	1	4
Perimeter and area	0	4
Adding up pieces	12	2

Table 2 shows a fairly clear picture of how productive each conceptualization was for each overall context. In the mathematics context, it appears that all three conceptualizations are roughly equally helpful and useful for students in *making sense* of the integral expressions in ways that were also satisfying to them. While there were certainly a few instances of confusion or explicitly incorrect statements, all three were largely productive, allowing for the conclusion that for decontextualized mathematics integrals, it really does not matter the lens through which a student decides to interpret the expression. All three can allow the student to *make sense* of the integral in question.

However, the physics context is where a striking difference can be seen. The adding up pieces conception was significantly productive in making sense of the applied integral expressions and equations. By contrast, most of the time when the students in this study invoked the anti-derivative or area under a curve conceptualizations, they were confused by their interpretations. The lone "productive" episode of the function matching symbolic form occurred only after Devon had already used the adding up pieces conception to fully explain the integral $\int_0^{600} R dt$. After he had discussed *what* the integral calculated and *why* it calculated it, he explained that an exact value could not be worked out by hand in its current form, since R was not specified as a function of t. Thus, coupled with an adding up pieces perspective, the anti-derivative notion adds some useful calculational insight. Similarly, once the adding up pieces conception is used to make sense of the integral, the area under a curve notion could be useful to get a sense of the magnitude of the resultant accumulated quantity. However, it appears that on their own, without adding up pieces, they lack the explanatory power that enables students to make meaningful interpretations of the integral expression.

Thus, these results demonstrate that in the physics context, the conceptualizations of area under a curve, anti-derivatives, and adding up pieces are *not* equal in productiveness. This allows for an answer to the third research question to be proposed. That is, the "overlap" in productiveness between the two contexts is that adding up pieces is generally productive in both the mathematics *and* the physics contexts. The "disjunction" in productiveness between the two contexts is that both the function matching and the perimeter and area symbolic forms, while productive in the mathematics context, were generally less productive in the physics context. In fact, adding up pieces was productive in so many instances that I label it *highly productive* for making sense of applied integral expressions and equations. Interestingly, the adding up pieces conceptualization was able to provide meaning to the area underneath the curve even when the perimeter and area conception itself could not provide this meaning to the students.

5. Discussion

The anti-derivative, area under a curve, and adding up pieces conceptualizations of the definite integral are all useful in certain contexts. Specifically, for decontextualized integrals, all three provide students with the cognitive tools to interpret definite integrals to their own satisfaction. Thus, we might hope that students leave their first-year calculus courses with *all three* conceptualizations cognitively ready for use. However, it appears that once students enter physics and engineering courses (and perhaps even subsequent mathematics courses), where they will see integrals of the types used in this study, not all three conceptualizations are equally useful in making sense of definite integrals.

5.1. The importance of the multiplicative relationship between integrand and differential

Perhaps the best recipe for understanding and working with integrals in physics and engineering courses might be described as a two-step process. First, students should interpret the definite integral through the adding up pieces lens, in order to analyze the multiplicative relationship between the integrand and the differential and the summation/accumulation of the resultant quantity. Once the students have done this to make sense of what the integral is calculating and *why*, then, second, the students can appeal to the area under a curve for analytic purposes or to an anti-derivative for calculational ease. Without this critical first step, however, this study suggests that students may be at a loss for understanding the physics and engineering integrals they encounter. In fact, understanding the multiplicative relationship between the integrand and the differential seems to have been the key to making sense of contextualized definite integrals for these students, so much so that, perhaps, a more descriptive name for this conceptualization is in order, since *adding up pieces* only highlights the *addition* aspect of the conceptualization. I propose the term *multiplicatively-based summation conception* as a more fitting name for what I have called the *adding up pieces* symbolic form in this and previous papers (see Jones, 2013). This emphasizes that the conception is one that incorporates both (a) the multiplicative relationship between the integrand and the differential to produce a "resultant quantity" and (b) the summation or accumulation of this resulting quantity throughout the domain prescribed by the bounds of integration. That is, the conception is not merely about addition, but is centered on this multiplicative relationship. I reiterate once again that this conception is not equivalent to the Riemann sum or the Riemann integral, yet these two main aspects of the adding up pieces, or rather the *multiplicatively-based summation conception*, are deeply rooted in the notion of the limit of Riemann sums.

I wish to note here that students may hold other conceptualizations of the definite integral, beyond what the students in this study exhibited, such as the "method of indivisibles" described in Section 2.1 from Czarnocha et al. (2001). The results of this paper may help provide a way for determining whether a given conceptualization, such as "indivisibles," might have productive power for helping students make sense of contextualized definite integrals. For example, while the method of indivisible might provide a student with a notion of how the area under a curve is "filled in" in a pure mathematics context, it is possible that the "straight lines" from this conception hide the critical multiplicative relationship between the integrand and differential. Each straight line in this conception consists merely of a single length represented by the integrand, which may be similar to the problematic "adding up the integrand" conception discussed in Jones (2013). In this way, we could see it is possible that students drawing on a conception like indivisibles might not perceive the underlying multiplicative relationship, which could potentially obscure the meaning of the applied integral expression.

5.2. Implications for curriculum

In light of the importance of the multiplicative and summative (or accumulative) ideas contained in the *multiplicatively-based summation conception*, we might expect to find that Riemann sums and the Riemann integral, on which the conception is based, play a significant role in shaping the conception of integration in calculus texts. Yet, when we consider "typical" textbooks (see Salas et al., 2006; Stewart, 2012; Thomas et al., 2009), there are indications this might not actually be the case. In the opening chapter on integration, these three calculus textbooks place a heavy emphasis on areas under curves and anti-derivatives. While all three textbooks do, of course, treat Riemann sums and use the Riemann integral definition, the Riemann sum is often portrayed only as a calculational method for computing irregular areas. That is, the Riemann sum usually appears only as a "tool" for getting at the "real" problem: finding areas underneath curves. This may cause the perimeter and area conceptualization to dominate students' perception of the definite integral's meaning. Perhaps it would serve students better to flip this around to highlight the *intrinsic* worth of the Riemann sum as an important conceptual entity in and of itself.

It is true that in the history of the calculus reform movement many attempts were made to shift the focus to the conceptual underpinnings of calculus topics (e.g., Dubinsky & Tall, 1991; Heid, 1988; Meel, 1998). Textbooks were developed to codify this conceptual learning into the curriculum, and while some texts did not endure the reaction against reform (Dubinsky, Schwingendorf, & Mathews, 1994), others continue to press on (Hughes-Hallett et al., 2012). In particular, in the reform movement the Riemann sum has become a greater focus in teaching integration (e.g., Hughes-Hallett et al., 2012; Thomas, 1995). Yet, while early reform efforts focused on deeper conceptual meaning, they did not always include as much attention to how other disciplines, like science and engineering, make use of calculus topics.

As a result of the increasing awareness of the mathematical needs of science and engineering students that inhabit calculus courses, a sort of "new reform" effort has emerged that specifically focuses on bridging the divide between calculus and science and engineering. Several recent curriculum projects have been attempted at various universities to blend together calculus and physics courses and workshops (Dray, Edwards, & Manogue, 2008; Hoffmann, 2004; Marrongelle, Black, & Meredith, 2003; More & Hill, 2002). Many support the notion of introducing calculus topics in a way that is directly applicable to physics. This study may hopefully serve as an additional piece of evidence for some of these efforts. While the results of this study are based off of a small sample size, it suggests the possibility that it may be desirable to highlight during instruction the multiplicative relationship between the integrand and the differential, and the summation or accumulation of the resulting quantity. Doing so might serve students in better making sense of definite integrals as well as how they are used in applied contexts. For example, even when physics texts use the notion of the "accumulated area under a graph" (see Serway & Jewett, 2008; Wilson et al., 2010), the multiplicative relationship is still required to even know *what* that accumulated area represents. The conclusion drawn from this paper are a small but hopefully useful step toward addressing the gap between how students learn integration in their mathematics courses and how they apply that knowledge to science (see Christensen & Thompson, 2010; Pollock, Thompson, & Mountcastle, 2007).

I wish to be clear that I am in no way advocating that we cease to teach areas nor anti-derivatives in calculus as important conceptualizations of the integral. These have worthwhile uses, since some integrals *can* be easily understood or analyzed through the geometric area of a shape under a curve, and since anti-derivatives provide the most efficient technique for *computing* integrals by hand. However, I advocate for an increased and *recurring* emphasis on the multiplicatively-based summation conception during the introductory lessons on integration in order to support the construction of this useful conceptualization. The findings of this paper align with others' conclusions that calculus instructors need to attend to how our largely science- and engineering-oriented students (Bressoud et al., 2013) use mathematics in their fields of study (see, for example, Bissell & Dillon, 2000; Gainsburg, 2007; Torigoe & Gladding, 2011). We should strive to adapt our classroom presentations to be consistent with how these students draw on mathematical notions and ideas in their respective fields of study (see Gainsburg, 2006).

6. Conclusion

In summary, the *adding up pieces* symbolic form, which has been renamed in this paper as the *multiplicatively-based summation conception*, is highly productive for making sense of applied integral expressions and equations. It helped the students in this study make sense of a variety of definite integrals, especially in places where the anti-derivative and area under a curve conceptualizations failed to. It allowed the students to utilize a representative rectangle (or square, or cube, or other shape) to understand how the integrand and the differential interact through multiplication. This in turn enabled them to explain how the integral works, in that each "little piece" has some small amount of the resultant quantity, and that these small amounts are added up to recapture the total amount. Thus, the multiplicatively-based summation conception is a highly useful and beneficial understanding that should be promoted in calculus curriculum and instruction.

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Appendix A. Interview items (in the order in which they were presented)

A.1. Mathematics-framed interview items

A.1.1. ITEM Math1

Two wires are attached to two telephone poles. Suppose we wanted to know the area between the two wires. How could you figure that out?

A.1.2. ITEM Math2

$\int_1^2 \frac{2}{x^3} - x^2 dx$ Compute and then discuss this integral. (Note: parentheses were omitted from this item in order to potentially generate useful discussion.)

A.1.3. ITEM Math3

Talk about what the following integral means.

$$\int_2^0 e^x dx$$

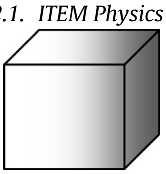
A.1.4. ITEM Math4

Suppose we had a function $f(x)$ with a domain D. What does this integral mean?

$$-2\int_D f(x)dx$$

A.2. Physics-framed interview items

A.2.1. ITEM Physics1



This shows a box with varying density (dark = more dense, light = less dense). Suppose you wanted to know the box's mass. How could you figure that out?

A.2.2. ITEM Physics2

The durability of a car motor is being tested. The engineers run the motor at *varying* levels of "revolutions per minute" over a 10 hour period. Denote "revolutions per minute" by R.

What is the *meaning* of the integral $\int_0^{600} Rdt$?

A.2.3. ITEM Physics3

A 2-dimensional surface (S) experiences a non-uniform pressure (P) and we want to know the total force exerted. We can use the surface's area (A) to compute this through the integral:

$F = \int_S PdA$. Why does this integral calculate the total force exerted?

A.2.4. ITEM Physics4

F_y is used to denote the amount of a force in the y-direction. ΔU is used to denote the change in potential energy. These two concepts are related through this equation:

$\Delta U = -\int_{y_i}^{y_f} F_y dy$. Explain this equation. What does each part of the equation/integral mean?

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